SUBJECT: MATHEMATICS

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Solution of Non-homogeneous Volterra Integral equation of second kind by successive approximation method.

A volterra integral equation of second kind

$$\phi(x) = F(x) + \lambda \int_0^x K(x, \xi)\phi(\xi)d\xi$$ has

one and only one solution, given by

$$\phi(x) = F(x) + \lambda \int_a^x R(x, \xi; \lambda) F(\xi) d\xi$$

where

$$R(x, \xi; \lambda) = K(x, \xi) + \sum_{\nu=1}^{\infty} \lambda^{\nu} K_{\nu+1}(x, \xi)$$

or, $$R(x, \xi; \lambda) = \sum_{\nu=1}^{\infty} \lambda^{\nu-1} K_{\nu}(x, \xi)$$

Proof. Consider the volterra integral equation of second kind

$$\phi(x) = F(x) + \lambda \int_0^x K(x, \xi)\phi(\xi)d\xi \quad \text{--- (1)}$$

where the kernel $K(x, \xi)$ is continuous in $]0, a[$ and $F(x)$ is continuous in $]0, a[$.

Let

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots + \lambda^n\phi_n(x) + \cdots \quad \text{--- (2)}$$

be a solution of (1) then we have

$$\phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots + \lambda^n\phi_n(x) + \cdots$$
$$= F(x) + \lambda\int_0^x K(x, \xi)\left[\phi_0(\xi) + \lambda\phi_1(\xi) + \lambda^2\phi_2(\xi) + \cdots + \lambda^n\phi_n(\xi) + \cdots\right] \quad \text{--- (3)}$$

On comparing we have

$$\phi_0(x) = F(x) \quad \text{———— (a1)}$$

$$\phi_1(x) = \int_0^x K(x,\xi)\,\phi_0(\xi)\,d\xi \quad \text{———— (a2)}$$

$$\phi_2(x) = \int_0^x K(x,\xi)\,\phi_1(\xi)\,d\xi \quad \text{———— (a3)}$$

$$\phi_3(x) = \int_0^x K(x,\xi)\,\phi_2(\xi)\,d\xi \quad \text{———— (a4)}$$

$$\equiv \equiv \equiv$$

$$\phi_n(x) = \int_0^x K(x,\xi)\,\phi_{n-1}(\xi)\,d\xi \quad \text{———— (a n+1)}$$

Equation (a2) $\Rightarrow$

$$\phi_1(x) = \int_0^x K(x,\xi)\,F(\xi)\,d\xi \quad \left[\because \phi_0(\xi) = f(\xi) \text{ from } (a_1)\right]$$

Now,

$$\phi_2(x) = \int_0^x K(x,\xi)\left[\int_0^\xi K(\xi,\xi_1)\,F(\xi_1)\,d\xi_1\right]d\xi \text{, from Just above}$$

Here $\xi_1 = 0$, $\xi_1 = \xi$ and $\xi = 0$, $\xi = x$

By interchanging the order of integration

$$\phi_2(x) = \int_{\xi_1=0}^{\xi_1=x} F(\xi_1)\left\{\int_{\xi=\xi_1}^{\xi=x} K(x,\xi)\,K(\xi,\xi_1)\,d\xi\right\}d\xi_1$$

Let $\displaystyle\int_{\xi=\xi_1}^{\xi=x} K(x,\xi)\,K(\xi,\xi_1)\,d\xi = K_2(x,\xi_1)$

then

$$\phi_2(x) = \int_0^x F(\xi_1)\cdot K_2(x,\xi_1)\,d\xi_1$$

$$\phi_2(x) = \int_0^x K_2(x,\xi_1)\,F(\xi_1)\,d\xi_1 \quad \text{———— (4)}$$

where $K_2(x,\xi_1) = \int K(x,\xi)\,K(\xi,\xi_1)\,d\xi$.

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or

$$\phi_2(x) = \int_0^x K_2(x,\xi) f(\xi) d\xi$$

⋮

In general

$$\phi_n(x) = \int_0^x K_n(x,\xi) f(\xi) d\xi$$

where $K_n(x,\xi) = \int_\xi^x K(x,z) K_{n-1}(z,\xi) dz$

Here, $K(x,z) = K_1(x,z)$

∴ solution(2) of (1) can be written as [(2) is $\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots$]

$$\phi(x) = f(x) + \lambda \int_0^x K(x,\xi) f(\xi) d\xi + \lambda^2 \int_0^x K_2(x,\xi) f(\xi) d\xi + \cdots + \lambda^n \int_0^x K_n(x,\xi) f(\xi) d\xi + \cdots$$

$$= f(x) + \lambda \left[\int_0^x \left\{ K(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \cdots \right\} f(\xi) d\xi \right]$$

$$\phi(x) = f(x) + \lambda \int_0^x R(x,\xi;\lambda) f(\xi) d\xi \qquad \left\{ \text{Here } K_1(x,\xi) = K(x,\xi) \right\}$$

which is the required solution of

$$\phi(x) = f(x) + \lambda \int_0^x K(x,\xi) \phi(\xi) d\xi$$

Example. Solve with the aid of resolvent Kernel of $\phi(x) = x + \int_0^x (\xi - x) \phi(\xi) d\xi$

Solution. Given $f(x) = x$, $K(x,\xi) = \xi - x$, $\lambda = 1$, Here

Here $K_1(x,\xi) = K(x,\xi) = (\xi - x)$

We know that

$$K_n(x,\xi) = \int_\xi^x K_1(x,z) K_{n-1}(z,\xi) dz$$

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$$\because K_2(x,\xi) = \int_{\xi}^{x} K_1(x,z)K_1(z,\xi)\,dz$$

$$= \int_{\xi}^{x} (z-x)(\xi-z)\,dz$$

$$= (z-x)(-)\frac{(\xi-z)^2}{\lfloor 2}\Big|_{\xi}^{x} - \int_{\xi}^{x} 1.(-)\frac{(\xi-z)^2}{\lfloor 2}\,dz$$

$$= 0 + \int_{\xi}^{x} \frac{(\xi-z)^2}{\lfloor 2}\,dz = -\frac{1}{\lfloor 3}(\xi-z)\Big|_{\xi}^{x}$$

$$K_2(x,\xi) = -\frac{1}{\lfloor 3}(\xi-x)^3$$

$$K_3(x,\xi) = \int_{\xi}^{x} K_1(x,z)K_2(z,\xi)\,dz$$

$$= \int_{\xi}^{x} (z-x)\left[-\frac{1}{\lfloor 3}(\xi-z)^3\right]dz$$

$$= -\frac{1}{\lfloor 3}\left[(z-x)(-)\frac{(\xi-z)^4}{4}\Big|_{\xi}^{x} - \int_{\xi}^{x} 1.(-)\frac{(\xi-z)^4}{4}\,dz\right]$$

$$= -\frac{1}{\lfloor 3}\left[0 + \int_{\xi}^{x} \frac{(\xi-z)^4}{4}\,dz\right]$$

$$K_3(x,\xi) = -\frac{1}{\lfloor 4}\left[(-)\frac{(\xi-z)^5}{5}\right]_{\xi}^{x} = \frac{1}{\lfloor 5}(\xi-x)^5$$

$$\therefore R(x,\xi;\lambda) = K_1(x,\xi) + K_2(x,\xi) + K_3(x,\xi) + \cdots$$

$$= (\xi-x) - \frac{(\xi-x)^3}{\lfloor 3} + \frac{(\xi-x)^5}{\lfloor 5} - \cdots$$

$$R(x,\xi;\lambda) = \sin(\xi-x) \quad \left[\because \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots\right]$$

, $\lambda = 1$, Here

We know solution of

$$\phi(x) = f(x) + \lambda\int_0^x K(x,\xi)\,\phi(\xi)\,d\xi \text{ is}$$

$$\phi(x) = f(x) + \lambda\int_0^x R(x,\xi;\lambda)\,f(\xi)\,d\xi$$

Required solution is

$$\therefore\ \phi(x) = x + 1.\int_0^x -\sin(\xi - x)\cdot(x)\,d\xi \quad x = \xi$$

$$= x + \int_0^x -\sin(\xi - x).\ \xi\,d\xi$$

$$= x + \left[\xi.\ \cos(\xi - x)\,(-)\Big|_{\xi=0}^{\xi=x} - \int_0^x (-)\cos(\xi - x)\,d\xi\right]$$

$$= \cancel{x} - \cancel{x} + \int_0^x \cos(\xi - x)\,d\xi$$

$$= \left[-\sin(\xi - x)\right]_0^x = -\sin(x - x) - \sin(0 - x)$$

$$\phi(x) = 0 - \sin(-x) = \sin x$$

$\Rightarrow\ \phi(x) = \sin x$ is the required solution of (1).

Q 2. Solve with the help of resolvent Kernel

$$\phi(x) = 1 + \int_0^x (\xi - x)\,\phi(\xi)\,d\xi$$

Soln Do as Q.N. 1, Here $f(x) = 1$, rest is same as Q.No.01.

i.e Here $R(x,\xi;\lambda) = \sin(\xi - x)$

$\therefore$ Req. solution will be

$$\phi(x) = 1 + \int_0^x -\sin(\xi - x).\,1\,d\xi$$

$$= 1 - 1 + \cos x = \cos x$$

$\therefore\ \phi(x) = \cos x$ is the solution of

$$\phi(x) = 1 + \int_0^x (\xi - x)\,\phi(\xi)\,d\xi.$$

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(3) Solve $\phi(x) = 1 + \int_0^x \phi(\xi)\,d\xi$

Solution. Here $\phi(x) = 1 + \int_0^x \phi(\xi)\,d\xi$ —— (1)

Here $f(x) = 1$, $K(x,\xi) = K_1(x,\xi) = 1$, $\lambda = 1$

$\Rightarrow f(\xi) = 1$

The Solution of (1) will be

$$\phi(x) = 1 + \int_0^x R(x,\xi;\lambda)\, f(\xi)\,d\xi \qquad \text{—— (2)}$$

where

$$R(x,\xi;1) = K_1(x,\xi) + K_2(x,\xi) + K_3(x,\xi) + \cdots \qquad (\text{as } \lambda = 1)$$

$\therefore$ $K_1(x,\xi) = 1$

$$K_2(x,\xi) = \int_\xi^x K_1(x,z)\,K_1(z,\xi)\,dz = \int_\xi^x 1.1\,dz = (x-\xi)$$

$$K_3(x,\xi) = \int_\xi^x K_1(x,z)\,K_2(z,\xi)\,dz = \int_\xi^x 1.(z-\xi) = \frac{(z-\xi)^2}{2}\Big|_\xi^x$$

$$K_3(x,\xi) = \frac{(x-\xi)^2}{\underline{|2}}$$

$-----$

$$\therefore R(x,\xi;1) = 1 + (x-\xi) + \frac{(x-\xi)^2}{\underline{|2}} + \cdots = e^{(x-\xi)}$$

So, Required Solution is

$$\phi(x) = 1 + \int_0^x e^{x-\xi}.1.\,d\xi = 1 + e^x\left[\frac{e^{-\xi}}{-1}\right]_0^x$$

$$\Rightarrow \phi(x) = 1 - e^x\left[e^{-x} - 1\right]$$

$$\Rightarrow \phi(x) = e^x$$

$\Rightarrow \phi(x) = e^x$ is the solution of $\phi(x) = 1 + \int_0^x \phi(\xi)\,d\xi$.

(4) Find the resolvent kernel of $K(x,\xi)=1$.

Solution. Given $K(x,\xi)=K_1(x,\xi)=1$

To find

$$R(x,\xi;\lambda)=K_1(x,\xi)+\lambda K_2(x,\xi)+\lambda^2 K_3(x,\xi)+\cdots$$

Here $K_1(x,\xi)=K(x,\xi)=1$

$$K_2(x,\xi)=\int_\xi^x K_1(x,z)K_1(z,\xi)\,dz=\int_\xi^x 1.1.dz=(x-\xi)$$

$$K_3(x,\xi)=\int_\xi^x K_1(x,z)K_2(z,\xi)\,dz=\int_\xi^x 1.(z-\xi)\,dz=\frac{(x-\xi)^2}{\underline{|2}}$$

$$\therefore R(x,\xi;\lambda)=1+\lambda(x-\xi)+\lambda^2\frac{(x-\xi)^2}{\underline{|2}}+\cdots=e^{(x-\xi)\lambda}$$

(5) Find $R(x,\xi;\lambda)$ of

$$K(x,\xi)=\begin{cases} e^{x-\xi}, & x>\xi \\ 0, & x<\xi \end{cases}$$

Solution. Given $K(x,\xi)=e^{x-\xi}$, $x>\xi$

To find $R(x,\xi;\lambda)=?$

We know that

$$R(x,\xi;\lambda)=K_1(x,\xi)+\lambda K_2(x,\xi)+\lambda^2 K_3(x,\xi)+\cdots$$

Here $K_1(x,\xi)=K(x,\xi)=e^{x-\xi}$

$$K_2(x,\xi)=\int_\xi^x K_1(x,z)K_1(z,\xi)\,dz=\int_\xi^x e^{x-z}.e^{z-\xi}\,dz$$

$$K_2(x,\xi)=\int_\xi^x e^{x-\xi}\,dz=e^{x-\xi}[z]_\xi^x=(x-\xi)e^{x-\xi}$$

$$K_3(x,\xi)=\int_\xi^x K_1(x,z)K_2(z,\xi)\,dz=\int_\xi^x e^{x-z}.(z-\xi)e^{z-\xi}\,dz$$

$$K_3(x,\xi)=e^{x-\xi}.\left.\frac{(z-\xi)^2}{\underline{|2}}\right|_\xi^x=\frac{(x-\xi)^2}{\underline{|2}}e^{x-\xi}$$

$\therefore R(x,\xi;\lambda) = e^{x-\xi} + \lambda(x-\xi)e^{x-\xi} + \frac{\lambda^2}{\lfloor 2}(x-\xi)^2 e^{x-\xi} + \cdots$

$= e^{(x-\xi)}\left[1 + \lambda(x-\xi) + \frac{\lambda^2}{\lfloor 2}(x-\xi)^2 + \cdots\right]$

$R(x,\xi;\lambda) = e^{(x-\xi)} \cdot e^{\lambda(x-\xi)} = e^{(1+\lambda)(x-\xi)}$

$\boxed{\text{i.e } R(x,\xi;\lambda) = e^{(1+\lambda)(x-\xi)}}$

(6) Find $R(x,\xi;\lambda)$ of $K(x,\xi) = \frac{2+\cos x}{2+\cos\xi}$

Solution. Given $K(x,\xi) = \frac{2+\cos x}{2+\cos\xi}$

To find $R(x,\xi;\lambda) = ?$

We know that

$R(x,\xi;\lambda) = K_1(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \cdots$

Here $K_1(x,\xi) = K(x,\xi) = \frac{2+\cos x}{2+\cos\xi}$

$K_2(x,\xi) = \int_\xi^x K_1(x,z)K_1(z,\xi)\,dz$

$K_2(x,\xi) = \int_\xi^x \frac{(2+\cos x)}{(2+\cos z)} \cdot \frac{(2+\cos z)}{(2+\cos\xi)}\,dz = (x-\xi)\cdot\left(\frac{2+\cos x}{2+\cos\xi}\right)$

$K_3(x,\xi) = \int_\xi^x K_1(x,z)K_2(z,\xi)\,dz$

$= \int_\xi^x \frac{2+\cos x}{(2+\cos z)}(z-\xi)\frac{(2+\cos z)}{2+\cos\xi}\,dz$

$K_3(x,\xi) = \frac{2+\cos x}{2+\cos\xi} \cdot \frac{(x-\xi)^2}{\lfloor 2}, \cdots$

$\therefore R(x,\xi;\lambda) = \frac{2+\cos x}{2+\cos\xi} + \lambda(x-\xi)\frac{2+\cos x}{2+\cos\xi} + \frac{\lambda^2}{\lfloor 2}(x-\xi)^2\frac{2+\cos x}{2+\cos\xi} + \cdots$

$R(x,\xi;\lambda) = \frac{2+\cos x}{2+\cos\xi}\left[1 + \lambda(x-\xi) + \frac{\lambda^2(x-\xi)^2}{\lfloor 2} + \cdots\right] = \frac{2+\cos x}{2+\cos\xi} \cdot e^{\lambda(x-\xi)}$

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Special method to find $R(x,\xi;\lambda)$ when $K(x,\xi)$ is a fn of $(x-\xi)$.

If $K(x,\xi)$ is a polynomial of degree $(n-1)$ in $(x-\xi)$ such that it may be expressed as

$$K(x,\xi) = a_0(x) + a_1(x)(x-\xi) + a_2(x)\frac{(x-\xi)^2}{\lfloor 2}$$
$$+ \cdots + \frac{a_{n-1}(x)}{\lfloor n-1}(x-\xi)^{n-1} \quad \text{--- (1)}$$

Let us auxiliary function be

$$\phi(x,\xi;\lambda) = \frac{(x-\xi)^{n-1}}{\lfloor n-1} + \lambda\int_\xi^x R(t,\xi;\lambda)\left[\frac{(x-\xi)^{n-1}}{\lfloor n-1}\right]_{\xi=t} dt$$

$$= \frac{(x-\xi)^{n-1}}{\lfloor n-1} + \lambda\int_\xi^x R(t,\xi;\lambda)\cdot\frac{1}{\lfloor n-1}(x-t)^{n-1}dt \quad \text{--- (2)}$$

with conditions
$$\left.\begin{array}{l}\phi = 0 = \frac{d\phi}{dx} = \frac{d^2\phi}{dx^2} = \cdots = \frac{d^{n-2}\phi}{dx^{n-2}} \text{ at } x=\xi\end{array}\right\} \text{--- (3)}$$

But $\frac{d^{n-1}\phi}{dx^{n-1}} = 1$ at $x=\xi$

In addition $R(x,\xi;\lambda) = \frac{1}{\lambda}\frac{d^n}{dx^n}\phi(x,\xi;\lambda)$ --- (4)

We know that

$$R(x,\xi;\lambda) = K(x,\xi) + \lambda\int_\xi^x R(z,\xi;\lambda)K(x,z)\,dz$$

[i.e $R(x,\xi;\lambda)$ satisfies the functional equation] --- (5)

Equ's (4) and (5) $\Rightarrow$

$$\frac{1}{\lambda}\frac{d^n}{dx^n}\phi = K(x,\xi) + \lambda\int_\xi^x R(z,\xi;\lambda)K(x,z)\,dz$$

$$\Rightarrow \frac{d^n\phi}{dx^n} = \lambda K(x,\xi) + \lambda^2\int_\xi^x \frac{d^n}{dz^n}\phi(z,\xi;\lambda).K(x,z)\,dz$$

$$\frac{d^n\phi}{dx^n} = \lambda K(x,\xi) + \lambda^2\left[K(x,z)\frac{d^{n-1}\phi}{dz^{n-1}} - \frac{\partial K}{\partial z}\cdot\frac{d^{n-2}\phi}{dz^{n-2}} + \cdots + \phi\cdot\frac{d^{n-1}K}{dz^{n-1}}\right]_{z=\xi}^{z=x}$$

$$\Rightarrow \frac{d^n\phi}{dx^n} = \lambda K(x,\xi) + 0$$

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$$\frac{d^n\phi}{dx^n} = \lambda\left[a_0(x) + a_1(x).(x-\xi) + a_2(x)\frac{(x-\xi)^2}{\lfloor 2} + \dots + a_{n-1}(x)\frac{(x-\xi)^{n-1}}{\lfloor n-1}\right] \quad (6)$$

But

$$\frac{d^{n-1}\phi}{dx^{n-1}} = 1, \quad (x-\xi) = \frac{d^{n-2}\phi}{dx^{n-2}}, \quad \frac{(x-\xi)^2}{\lfloor 2} = \frac{d^{n-3}\phi}{dx^{n-3}}$$

$$\because \phi = \frac{1}{\lfloor n-1}(x-\xi)^{n-1} + \lambda\int_\xi^x R(t,\xi;\lambda)\frac{(x-t)^{n-1}}{\lfloor n-1}\,dt$$

$$\frac{d^{n-1}\phi}{dx^{n-1}} = \lfloor n-1 \cdot \frac{1}{\lfloor n-1} + 0$$ as second term has degree less than $(n-1)$.

$$\Rightarrow \frac{d^{n-1}\phi}{dx^{n-1}} = 1$$

Integrating from ξ to x

$$\frac{d^{n-2}\phi}{dx^{n-2}} = \int_\xi^x d\xi = (x-\xi)$$

Again, integrating

$$\frac{d^{n-3}\phi}{dx^{n-3}} = \frac{(x-\xi)^2}{\lfloor 2} \quad \dots$$

(6) equ $\Rightarrow$ $$\frac{d^n\phi}{dx^n} - \lambda\left[a_0(x).\frac{d^{n-1}\phi}{dx^{n-1}} + a_1(x).\frac{d^{n-2}\phi}{dx^{n-2}} + \dots + a_{n-1}(x)\phi\right] = 0. \quad (7)$$

So, Remember $$R(x,\xi;\lambda) = \frac{1}{\lambda}\frac{d^n}{dx^n}\phi, \quad (8)$$

and $$\phi = \frac{d\phi}{dx} = \dots = \frac{d^{n-2}}{dx^{n-2}} \ \& \ \frac{d^{n-1}\phi}{dx^{n-1}} = 1 \text{ at } x=\xi \quad (9)$$

and

$$K(x,\xi) = a_0(x) + (x-\xi)a_1(x) + \dots + \frac{(x-\xi)^{n-1}}{\lfloor n-1}a_{n-1}(x) \quad (10)$$

Example on next page

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(1) Find the resolvent kernel of $K(x, \xi) = 2x$, $\lambda = 1$

Soln Given $K(x, \xi) = 2x$ ——— (1)

We know that

$$K(x,\xi) = a_0(x) + (x-\xi)\,a_1(x) + \cdots + \frac{(x-\xi)^{n-1}}{\lfloor n-1}\,a_{n-1}(x) \quad \text{—— (2)}$$

comparing (1) and (2)

$$a_0(x) = 2x \quad \text{i.e.} \quad n-1 = 0 \Rightarrow n = 1$$

$\therefore$ $$\frac{d^n\phi}{dx^n} - \lambda\left[a_0(x)\frac{d^{n-1}\phi}{dx^{n-1}} + a_1(x)\frac{d^{n-2}\phi}{dx^{n-2}} + \cdots + a_{n-1}(x)\phi\right] = 0$$

$$\Rightarrow \frac{d\phi}{dx} - \lambda a_0(x) = 0$$

$$\Rightarrow \frac{d\phi}{dx} - 2x\lambda\phi = 0 \Rightarrow \frac{d\phi}{dx} - 2x\phi = 0 \quad , \phi = 1 \text{ at } x = \xi$$

$$\Rightarrow d\phi = 2x\,dx\,\phi$$

$$\Rightarrow \frac{d\phi}{\phi} = 2x\,dx \Rightarrow \log\phi = x^2 + \log c$$

$$\Rightarrow \log\frac{\phi}{c} = x^2 \Rightarrow \phi = c \cdot e^{x^2}$$

put $\phi = 1$, $x = \xi$

$$\therefore 1 = c \cdot e^{\xi^2} \Rightarrow c = e^{-\xi^2}$$

$$\therefore \phi = e^{-\xi^2} \cdot e^{x^2} \Rightarrow \phi = e^{x^2 - \xi^2}$$

$$\therefore R(x,\xi;\lambda) = \frac{1}{\lambda}\frac{d\phi}{dx} = \frac{1}{1}\frac{d}{dx}\left[e^{x^2} \cdot e^{-\xi^2}\right]$$

$$\Rightarrow R(x,\xi;\lambda) = e^{-\xi^2} \cdot 2x e^{x^2} = 2x\,e^{x^2-\xi^2}$$

$$\therefore \underline{R(x,\xi;1) = 2x e^{x^2-\xi^2}}$$

(2) Find $R(x,\xi;\lambda)$ of $K(x,\xi) = 2-(x-\xi)$, $\lambda = 1$

Soln Given $K(x,\xi) = 2-(x-\xi)$ ——— (1)

We know that

$$K(x,\xi) = a_0(x) + (x-\xi)\,a_1(x) + \frac{(x-\xi)^2}{\lfloor 2}a_2(x) + \cdots + \frac{(x-\xi)^{n-1}}{\lfloor n-1}a_{n-1}(x) \quad \text{—— (2)}$$

comparing (1) & (2), we get

$$a_0(x) = 2, \quad a_1(x) = -1 \quad \& \quad n-1 = 1 \Rightarrow n = 2$$

$\therefore$ Required diff eqn is

$$\frac{d^2\phi}{dx^2} - \lambda\left[a_0(x)\frac{d\phi}{dx} + a_1(x)\phi\right] = 0$$

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with conditions $\frac{d\phi}{dx} = 1$ at $x = \xi$

& $\phi = 0$ at $x = \xi$

$\therefore \frac{d^2\phi}{dx^2} - \lambda\left[a_0(x)\frac{d\phi}{dx} + a_1(x)\phi\right] = 0$

$\Rightarrow \frac{d^2\phi}{dx^2} - \left[2\frac{d\phi}{dx} + (-1)\phi\right] = 0$

$\Rightarrow \frac{d^2\phi}{dx^2} - 2\frac{d\phi}{dx} + \phi = 0$ — (3) with conditions

$\left.\begin{array}{l}\phi = 0 \text{ at } x = \xi \\ \frac{d\phi}{dx} = 1 \text{ at } x = \xi\end{array}\right\}$ — (4)

Its auxiliary equation is

$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore$ Solution of (3) is given by

$\phi = (A + Bx)e^x$ — (5)

put $\phi = 0$, $x = \xi$

$\Rightarrow 0 = (A + B\xi)e^\xi \Rightarrow A + B\xi = 0$ — (6)

Now $\frac{d\phi}{dx} = Be^x + (A+Bx)e^x$

$\left(\frac{d\phi}{dx}\right)_{x=\xi} = Be^\xi + (A+B\xi)e^\xi$

$\Rightarrow 1 = Be^\xi$ $[\because A + B\xi = 0]$

$\Rightarrow B = e^{-\xi}$ — (7)

(6) & (7) $\Rightarrow A + B\xi = 0 \Rightarrow B\xi = -A$

$\Rightarrow A = -B\xi = -\xi e^{-\xi}$

$\therefore \phi = (A + Bx)e^x$

$\Rightarrow \phi = (-\xi e^{-\xi} + e^{-\xi} \cdot x)e^x \Rightarrow \phi = (x-\xi)e^{x-\xi}$

$\therefore R(x,\xi;1) = \frac{d^2\phi}{dx^2} = \frac{d^2}{dx^2}(x-\xi)e^{x-\xi}$

$R(x,\xi;1) = \frac{d}{dx}\left[(x-\xi)e^{x-\xi} + e^{x-\xi}\right] = \left[(x-\xi)e^{x-\xi} + e^{x-\xi} + e^{x-\xi}\right]$

Q for Practice: Find $R(x,\xi;1)$ for $K(x,\xi) = -2 + 3(x-\xi)$ ... Ans.